

# New Relativistic Wave Equations for Two-Particle Systems

Guang-Qing Bi <sup>†</sup>, Yue-Kai Bi

We seek to introduce a mathematical method to derive relativistic wave equations for two-particle systems. According to this method, if we define stationary wave functions as special solutions like  $\Psi(\mathbf{r}_1, \mathbf{r}_2, t) = \psi(\mathbf{r}_1, \mathbf{r}_2)e^{-iEt/\hbar}$ , and properly define the relativistic reduced mass  $\mu_0$ , then the new relativistic two-body wave equations can be derived. The spin-orbit coupling term is also derived from the relativistic correction terms. On this basis, we obtain the relativistic energy levels of ponium and pionic hydrogen atoms, using which, we discuss the pair production and annihilation of  $\pi^+$  and  $\pi^-$ , and find a reasonable explanation for the dark matter.

**Keywords:** Relativistic two-body wave equations, Bound state, Relativistic energy levels, Dark matter

**PACS:** 03.65.Pm, 31.15.aj, 02.30.Tb, 03.65.Ge

## 1. Introduction

Both special relativity and experiments indicate that, the mass of a many-particle system in a bound state is less than the sum of the rest mass of every particle forming the system, and the difference gives the mass defect of the system, while the product of the mass defect and the square of the speed of light gives the binding energy of the system. As the binding energy is quantized, the sum of it and the rest mass of every particle forming the system is the energy level of the system. For instance, the mass of an atomic nucleus is obviously less than the sum of the rest mass of every nucleon forming the atomic nucleus. Therefore, in order to express the mass defect explicitly, there is a necessity to introduce the concept of system mass, which differs from the sum of the rest mass of every particle forming the system. By introducing the concept of the system mass and applying proper mathematical skills, the relativistic wave equations for two-particle systems is derived. On this basis, let us properly define the relativistic reduced mass to further derive the new relativistic two-body wave equations.

The main results of this paper are expressed as

---

<sup>†</sup>Corresponding author. E-mail: guangqingbi@yahoo.cn

1. The relativistic two-body wave equations

$$\begin{aligned}
E'\psi = & -\frac{2(m_{01}\mu + m_{02}\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{\hbar^2}{m_{01}} \nabla_1^2 \psi - \frac{2(m_{02}\mu + m_{01}\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{\hbar^2}{m_{02}} \nabla_2^2 \psi \\
& + \frac{2\mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \psi \\
& - \frac{1}{(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right)^2 \psi \\
& + \frac{\hbar^2}{(m_0 + m)(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) (\nabla_1^2 + \nabla_2^2) \psi \\
& + \frac{\hbar^2}{(m_0 + m)(\mu_0 + \mu)c^2} (\nabla_1^2 + \nabla_2^2) \left[ \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \psi \right] \\
& - \frac{\hbar^4}{(m_0 + m)^2(\mu_0 + \mu)c^2} (\nabla_1^2 - \nabla_2^2)^2 \psi.
\end{aligned}$$

2. In the center-of-momentum frame, it is simplified as

$$\begin{aligned}
E'\psi = & -\frac{4\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)} \left( m - \frac{U}{c^2} \right)^2 \nabla^2 \psi \\
& + \frac{2\mu}{(\mu_0 + \mu)(m_0 + m)} \left( 2mU - \frac{U^2}{c^2} \right) \psi \\
& + \frac{2\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left[ \nabla^2 \left( 2mU - \frac{U^2}{c^2} \right) \right] \psi \\
& + \frac{4\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \nabla \left( 2mU - \frac{U^2}{c^2} \right) \cdot \nabla \psi \\
& - \frac{1}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( 2mU - \frac{U^2}{c^2} \right)^2 \psi.
\end{aligned}$$

Where  $m_0 = m_{01} + m_{02}$ ,  $E = mc^2$ , and  $m, \mu_0, \mu$  respectively denote

$$m = m_0 + \frac{1}{c^2} E', \quad \mu_0 = \frac{2m_{01}m_{02}}{m_0 + m}, \quad \mu = \mu_0 + \frac{1}{c^2} E'.$$

3. If  $U = -Ze_s^2/r$ ,  $e_s = e(4\pi\epsilon_0)^{-1/2}$ ,  $r = |\mathbf{r}_1 - \mathbf{r}_2|$  and  $E' < 0$ , then

$$E_n = \left[ m_{01}^2 \pm 2m_{01}m_{02} \left( 1 + \frac{Z^2\alpha^2}{(n - \sigma_l)^2} \right)^{-1/2} + m_{02}^2 \right]^{1/2} c^2.$$

$$\sigma_l = l + \frac{1}{2} + \frac{d_0}{2(n - \sigma_l)} - \sqrt{\left( l + \frac{1}{2} \right)^2 - Z^2\alpha^2 + \frac{3d_0}{2} - \frac{d_0}{n - \sigma_l}}, \quad \alpha = \frac{e_s^2}{\hbar c}.$$

$$d_0 = 2Z^2\alpha^2 D, \quad D = \frac{\mu(m_0 + m)}{2m^2}, \quad m = E_n/c^2, \quad \mu = \pm\mu_0 \left( 1 + \frac{Z^2\alpha^2}{(n - \sigma_l)^2} \right)^{-1/2}.$$

4. For spin 1/2 particles in the central field, the relativistic two-body wave equations is expressed as

$$\begin{aligned}
E'\psi = & -\frac{2(m_{01}\mu + m_{02}\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{\hbar^2}{m_{01}} \nabla_1^2 \psi - \frac{2(m_{02}\mu + m_{01}\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{\hbar^2}{m_{02}} \nabla_2^2 \psi \\
& + \frac{2\mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \psi \\
& + \frac{4\hbar^2(m - U/c^2)}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( \frac{dU}{dr_1} \frac{\partial \psi}{\partial r_1} + \frac{dU}{dr_2} \frac{\partial \psi}{\partial r_2} \right) \\
& + \frac{2\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( 2mU - \frac{U^2}{c^2} \right) (\nabla_1^2 + \nabla_2^2) \psi \\
& + \frac{\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left[ (\nabla_1^2 + \nabla_2^2) \left( 2mU - \frac{U^2}{c^2} \right) \right] \psi \\
& - \frac{1}{(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right)^2 \psi \\
& - \frac{8(m - U/c^2)}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( \frac{1}{r_1} \frac{dU}{dr_1} \mathbf{S}_1 \cdot \mathbf{L}_1 \psi + \frac{1}{r_2} \frac{dU}{dr_2} \mathbf{S}_2 \cdot \mathbf{L}_2 \psi \right) \\
& - \frac{\hbar^4}{(m_0 + m)^2(\mu_0 + \mu)c^2} (\nabla_1^2 - \nabla_2^2)^2 \psi.
\end{aligned}$$

Where  $\mathbf{L}_1$  is the orbital angular momentum of the first particle, and  $\mathbf{L}_2$  is that of the second one.

5. In the center-of-momentum frame, it is simplified as

$$\begin{aligned}
E'\psi = & -\frac{4\hbar^2}{(\mu_0 + \mu)(m_0 + m)^2} \left( m - \frac{U}{c^2} \right)^2 \nabla^2 \psi \\
& + \frac{2\mu}{(\mu_0 + \mu)(m_0 + m)} \left( 2mU - \frac{U^2}{c^2} \right) \psi \\
& + \frac{8\hbar^2(m - U/c^2)}{(m_0 + m)^2(\mu_0 + \mu)c^2} \frac{dU}{dr} \frac{\partial \psi}{\partial r} \\
& + \frac{2\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left[ \nabla^2 \left( 2mU - \frac{U^2}{c^2} \right) \right] \psi \\
& - \frac{1}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( 2mU - \frac{U^2}{c^2} \right)^2 \psi \\
& - \frac{8(m - U/c^2)}{(m_0 + m)^2(\mu_0 + \mu)c^2} \frac{1}{r} \frac{dU}{dr} \mathbf{S} \cdot \mathbf{L} \psi.
\end{aligned}$$

Where  $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ ,  $\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2$  be the total spin angular momentum and total orbital angular momentum operators of the two-particle system respectively,  $U(\mathbf{r}) = U(\mathbf{r}_1 - \mathbf{r}_2)$  be the interaction energy between particles.

## 2. Relativistic Two-body Wave Equations and Probability Meaning of Their Stationary Solutions

As we know, for a free particle, its energy and momentum are both constant, therefore, it is quite natural to assume that its matter wave is a plane wave. According to the de Broglie relation

$$\hbar \mathbf{k} = \mathbf{p}, \quad E = \hbar \omega,$$

we have the wave function of a free particle:

$$\Psi_p(\mathbf{r}, t) = A \exp(-i(Et - \mathbf{p} \cdot \mathbf{r})/\hbar). \quad (1)$$

Where  $E$  is the total energy of the particle containing the intrinsic energy  $m_0 c^2$ , and  $\mathbf{p}$  is the momentum of the particle.

The quantization method of quantum mechanics is assuming  $E$  and  $\mathbf{p}$  are correspondingly equivalent to the following two differential operators:

$$E \rightarrow i\hbar \frac{\partial}{\partial t}, \quad \mathbf{p} \rightarrow -i\hbar \nabla.$$

This relation is obviously tenable for the wave function of a free particle, while arbitrary wave function is equal to the linear superposition of the plane waves of free particles with all possible momentum, namely

$$\begin{aligned} \Psi(\mathbf{r}, t) &= \int \int \int_{-\infty}^{\infty} c(\mathbf{p}, t) \Psi_p(\mathbf{r}, t) dp_x dp_y dp_z, \\ c(\mathbf{p}, t) &= \int \int \int_{-\infty}^{\infty} \Psi(\mathbf{r}, t) \Psi_p^*(\mathbf{r}, t) dx dy dz. \end{aligned} \quad (2)$$

Where  $\Psi_p(\mathbf{r}, t)$  is expressed by (1),  $A = (2\pi\hbar)^{-3/2}$ . Therefore, this quantization method is also tenable for arbitrary wave functions. Let  $K$  and  $K'$  be two inertial systems, when  $K$  is transformed to  $K'$ , the difference between  $(E't' - \mathbf{p}' \cdot \mathbf{r}')/\hbar$  and  $(Et - \mathbf{p} \cdot \mathbf{r})/\hbar$  is at most an integral multiple of  $2\pi$ , and  $(Et - \mathbf{p} \cdot \mathbf{r})/\hbar$  is a relativistic invariant because of the phase periodicity. Then we have

$$A \exp[-i(Et - \mathbf{p} \cdot \mathbf{r})/\hbar] = A \exp[-i(E't' - \mathbf{p}' \cdot \mathbf{r}')/\hbar] \quad \text{or} \quad \Psi_p(\mathbf{r}, t) = \Psi_{p'}(\mathbf{r}', t').$$

Thus there is no contradiction between (2) and relativity, that is, this quantization method itself is relativistic. Clearly, (2) are the Fourier transform and its inversion, which can be extended to many-particle systems. For two-particle systems, let  $\mathbf{r}_1 = (x_1, y_1, z_1)$

and  $\mathbf{r}_2 = (x_2, y_2, z_2)$  be position vectors of two particles in the laboratory reference frame respectively, and corresponding momentum vectors be  $\mathbf{p}_1 = (p_{x_1}, p_{y_1}, p_{z_1})$  and  $\mathbf{p}_2 = (p_{x_2}, p_{y_2}, p_{z_2})$ . Thus related physical quantities in (1) and (2) are extended to  $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2)$ ,  $\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2)$ ,  $A = (2\pi\hbar)^{-3}$ ,  $dp_x = dp_{x_1}dp_{x_2}$ ,  $dp_y = dp_{y_1}dp_{y_2}$ ,  $dp_z = dp_{z_1}dp_{z_2}$ ,  $dx = dx_1dx_2$ ,  $dy = dy_1dy_2$ ,  $dz = dz_1dz_2$ . Therefore this quantization method can also be applied to many-particle systems. Then why the Schrödinger equation derived from this method is non-relativistic? That is because the relation between  $E$  and  $\mathbf{p}$  is non-relativistic, which is due to the fact that the kinetic energy is non-relativistic. Therefore, if we want to establish the relativistic wave equations, we need to introduce the relativistic kinetic expression.

Assuming that any particle with the rest mass  $m_0$ , no matter how high the speed is, no matter it is in a potential field or in free space, and no matter how it interacts with other particles, its kinetic energy is:

$$E_k = (c^2p^2 + m_0^2c^4)^{1/2} - m_0c^2. \quad (3)$$

This assumption has already been verified (See arXiv:1008.4224). On this basis, we can establish the relation between the system energy  $E$  and the momentum  $\mathbf{p}$  using proper mathematical skills, thus obtain the relativistic Hamiltonian. Therefore, we introduce the mathematical method in Reference [1]-[9]. Power functions and exponential functions play special roles in this method, which are called the base functions as we can establish mapping relations between them and arbitrary functions in a certain range. For instance, in quantum mechanics, (2) determines the mapping relations between wave functions of free particles and arbitrary wave functions. Similarly to Reference [1] or abstract operators (See arXiv:1008.3808), we can introduce the base function and relevant concepts in quantum mechanics:

**Definition 1** The right-hand side of (1) is defined as the base function of quantum mechanics, where  $E$  and  $\mathbf{p}$  are called the characters of base functions, while  $E$  and  $\mathbf{p}$  are not only suitable for free particles, but also suitable for any system, and the relation between  $E$  and  $\mathbf{p}$  is called the characteristic equation of wave equations. Different system has different characteristic equations.

According to differential laws, we have

$$i\hbar \frac{\partial}{\partial t} \Psi_p = E \Psi_p, \quad -i\hbar \nabla \Psi_p = \mathbf{p} \Psi_p. \quad (4)$$

**Definition 2** Let  $m_0 = m_{01} + m_{02} + \cdots + m_{0N}$  be the total rest mass of an  $N$ -particle system,  $E'$  be the sum of the kinetic energy and potential energy of all the  $N$  particles, then the actual mass of the system, which is called the system mass, is defined as

$$m = m_0 + \frac{1}{c^2} E'. \quad (5)$$

**Definition 3** If the system is in a bound state ( $E' < 0$ ), then the absolute value of  $E'$  is

$$|E'| = m_0c^2 - mc^2 = \Delta mc^2,$$

which is called the binding energy of the system, where  $\Delta m = m_0 - m$  is the mass defect of the system.

**Definition 4** The total energy of the system  $E$  is defined as the sum of the rest energy, kinetic energy and potential energy of all the particles forming the system, namely  $E = m_0c^2 + E'$

According to Definition 2 and 4, the total energy of the system is equal to the product of the system mass and the square of the speed of light, namely  $E = mc^2$ , thus the system mass is uniquely determined by the energy levels of the system.

**Definition 5** In relativistic quantum mechanics, the stationary wave function for two-particle system is defined as the following special solution:

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, t) = \psi(\mathbf{r}_1, \mathbf{r}_2) \exp(-iEt/\hbar). \quad (6)$$

Where  $E$  is the total energy of the two-particle system.

Applying (2) to Definition 5, we have

$$\begin{aligned} \psi(\mathbf{r}_1, \mathbf{r}_2) &= \frac{1}{(2\pi\hbar)^3} \int \int \int_{-\infty}^{\infty} c(\mathbf{p}_1, \mathbf{p}_2) \exp\left(\frac{i}{\hbar}(\mathbf{p}_1 \cdot \mathbf{r}_1 + \mathbf{p}_2 \cdot \mathbf{r}_2)\right) dp_x dp_y dp_z, \\ c(\mathbf{p}_1, \mathbf{p}_2) &= \frac{1}{(2\pi\hbar)^3} \int \int \int_{-\infty}^{\infty} \psi(\mathbf{r}_1, \mathbf{r}_2) \exp\left(-\frac{i}{\hbar}(\mathbf{p}_1 \cdot \mathbf{r}_1 + \mathbf{p}_2 \cdot \mathbf{r}_2)\right) dx dy dz. \end{aligned} \quad (7)$$

Where  $\mathbf{r}_1 = (x_1, y_1, z_1)$  and  $\mathbf{r}_2 = (x_2, y_2, z_2)$  are position vectors of two particles in the laboratory reference frame respectively, and  $dx = dx_1 dx_2$ ,  $dy = dy_1 dy_2$ ,  $dz = dz_1 dz_2$ . Corresponding momentum vectors are respectively  $\mathbf{p}_1 = (p_{x_1}, p_{y_1}, p_{z_1})$  and  $\mathbf{p}_2 = (p_{x_2}, p_{y_2}, p_{z_2})$ , and  $dp_x = dp_{x_1} dp_{x_2}$ ,  $dp_y = dp_{y_1} dp_{y_2}$ ,  $dp_z = dp_{z_1} dp_{z_2}$ .

In relativistic quantum mechanics, due to the relativistic effect that mass varies with speed, the center of mass system is no longer a proper description framework, instead, we use the center-of-momentum frame, which is a coordinate system that the total momentum equals zero. If  $v_1$ ,  $v_2$  respectively denote the speed of particles in the two-particle system, then their momentum respectively are

$$\mathbf{p}_1 = m_1 \mathbf{v}_1 = \frac{m_{01} \mathbf{v}_1}{\sqrt{1 - (v_1/c)^2}}, \quad \mathbf{p}_2 = m_2 \mathbf{v}_2 = \frac{m_{02} \mathbf{v}_2}{\sqrt{1 - (v_2/c)^2}},$$

and  $\mathbf{p}_1 = -\mathbf{p}_2$ . If  $v$  denotes the relative speed between two particles, then we can properly define the relativistic reduced mass  $\mu$  to make the relative momentum  $\mathbf{p} = \mu \mathbf{v}$

satisfy  $|\mathbf{p}_1| = |\mathbf{p}_2| = |\mathbf{p}|$ , namely

$$p_{x_1}^2 + p_{y_1}^2 + p_{z_1}^2 = p_{x_2}^2 + p_{y_2}^2 + p_{z_2}^2 = p_x^2 + p_y^2 + p_z^2. \quad (8)$$

In other words, the reduced mass  $\mu$  can be determined using (8) and the relativistic velocity addition formula. As it is related to speed, in order to distinguish it from another type of reduced mass, we call this one the speed-type reduced mass. For instance, if two particles of a two-particle system are restricted to movement along the same line, then its speed-type reduced mass is defined as

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \left( 1 + \frac{v_1 v_2}{c^2} \right). \quad (9)$$

Where

$$m_1 = \frac{m_{01}}{\sqrt{1 - (v_1/c)^2}}, \quad m_2 = \frac{m_{02}}{\sqrt{1 - (v_2/c)^2}}.$$

Therefore, using the center-of-momentum frame in (7),  $\mathbf{p}_2 = -\mathbf{p}_1$ ,  $|\mathbf{p}_1| = |\mathbf{p}|$ . Substituting them into (7), we have

$$\begin{aligned} \psi(\mathbf{r}_1, \mathbf{r}_2) &= \frac{1}{(2\pi\hbar)^3} \int \int \int_{-\infty}^{\infty} c(\mathbf{p}_1, \mathbf{p}_2) \exp(i\mathbf{p} \cdot (\mathbf{r}_1 - \mathbf{r}_2)/\hbar) dp_x dp_y dp_z, \\ c(\mathbf{p}_1, \mathbf{p}_2) &= \frac{1}{(2\pi\hbar)^3} \int \int \int_{-\infty}^{\infty} \psi(\mathbf{r}_1, \mathbf{r}_2) \exp(-i\mathbf{p} \cdot (\mathbf{r}_1 - \mathbf{r}_2)/\hbar) dx dy dz. \end{aligned} \quad (10)$$

Where  $|\mathbf{p}|$  is the relative momentum of the two-particle system. Relative coordinate is denoted by  $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ , then the result can be expressed by a more symmetric form

$$\begin{aligned} \Psi(\mathbf{r}_1, \mathbf{r}_2, t) &= \frac{1}{(2\pi\hbar)^3} \int \int \int_{-\infty}^{\infty} c(\mathbf{p}_1, \mathbf{p}_2) \exp(-i(Et - \mathbf{p} \cdot \mathbf{r})/\hbar) dp_x dp_y dp_z, \\ c(\mathbf{p}_1, \mathbf{p}_2) &= \frac{1}{(2\pi\hbar)^3} \int \int \int_{-\infty}^{\infty} \Psi(\mathbf{r}_1, \mathbf{r}_2, t) \exp(i(Et - \mathbf{p} \cdot \mathbf{r})/\hbar) dx dy dz. \end{aligned} \quad (11)$$

Where  $\Psi(\mathbf{r}_1, \mathbf{r}_2, t) = \psi(\mathbf{r}_1, \mathbf{r}_2) \exp(-iEt/\hbar)$  is the relativistic stationary wave functions for the two-particle system. Therefore,  $\mathbf{p} = \mu\mathbf{v}$ , which is the relative momentum of the two particle system in the center-of-momentum frame, is definitely equivalent to the differential operator  $-i\hbar\nabla$  with respect to the relative coordinate  $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ , still denoted by  $\mathbf{p} \rightarrow -i\hbar\nabla$ . This quantization rule is tenable for arbitrary stationary wave functions for two-particle systems.

Considering an isolated two-particle system, if the interaction energy between two particles is denoted by  $U(\mathbf{r}_1, \mathbf{r}_2)$ , then according to Definition 2 and (3), we have

$$E' - U = (c^2 p_1^2 + m_{01}^2 c^4)^{1/2} - m_{01} c^2 + (c^2 p_2^2 + m_{02}^2 c^4)^{1/2} - m_{02} c^2. \quad (12)$$

Where  $m_{01}$ ,  $m_{02}$  are the rest mass of two particles respectively, and corresponding momentum are  $p_1 = |\mathbf{p}_1|$ ,  $p_2 = |\mathbf{p}_2|$ . This is the characteristic equation of relativistic wave equations for the two-particle system, thus we obtain.

$$[\sqrt{c^2 p_1^2 + m_{01}^2 c^4} + \sqrt{c^2 p_2^2 + m_{02}^2 c^4} + U]\psi = E\psi. \quad (13)$$

This is the spin-less Salpeter equation (See [6], [7]), it is an important relativistic two-body wave equation. In order to make it easier to solve the corresponding relativistic wave equation, the characteristic equation (12) should be transformed to remove the fractional power, then we have

$$\begin{aligned} (E' - U + m_{01} c^2 + m_{02} c^2)^2 &= c^2 p_1^2 + m_{01}^2 c^4 + c^2 p_2^2 + m_{02}^2 c^4 \\ &\quad + 2(c^2 p_1^2 + m_{01}^2 c^4)^{1/2} (c^2 p_2^2 + m_{02}^2 c^4)^{1/2}. \end{aligned} \quad (14)$$

Expanding the left-hand side of (14) and applying (5), we have

$$\begin{aligned} &(m_0 + m)E' - 2mU + U^2/c^2 \\ &= p_1^2 + p_2^2 + 2(p_1^2 + m_{01}^2 c^2)^{1/2} (p_2^2 + m_{02}^2 c^2)^{1/2} - 2m_{01} m_{02} c^2. \end{aligned} \quad (15)$$

Further, removing the radical sign, we have

$$\begin{aligned} &[(m_0 + m)E' + 2m_{01} m_{02} c^2 - p_1^2 - p_2^2 - 2mU + U^2/c^2]^2 \\ &= 4(p_1^2 + m_{01}^2 c^2)(p_2^2 + m_{02}^2 c^2). \end{aligned} \quad (16)$$

**Definition 6** In relativistic quantum mechanics, a type of relativistic reduced mass  $\mu_0$  of two-particle systems is defined as

$$\mu_0 = \frac{2m_{01} m_{02}}{m_0 + m}, \quad \mu = \mu_0 + \frac{1}{c^2} E'. \quad (17)$$

Where  $m_0 = m_{01} + m_{02}$ ,  $m$  is the system mass of the two-particle system,  $E'$  is the sum of kinetic energy and potential energy of the two particles,  $\mu$  is called the system mass corresponding to  $\mu_0$ . Unless otherwise stated, the reduced mass referred in our paper from now on is defined in this way, which should not be confused with the speed-type reduced mass mentioned previously.

According to (17), we have

$$(m_0 + m)E' + 2m_{01} m_{02} c^2 = (m_0 + m)\mu c^2,$$



$$((m_0 + m)E' + 2m_{01}m_{02}c^2)^2 = (m_0 + m)^2(\mu_0 + \mu)c^2E' + 4m_{01}^2m_{02}^2c^4.$$

Thus (16) can be expressed as

$$\begin{aligned} & 4(p_1^2p_2^2 + m_{02}^2c^2p_1^2 + m_{01}^2c^2p_2^2) \\ = & (m_0 + m)^2(\mu_0 + \mu)c^2E' - 2(m_0 + m)\mu c^2(p_1^2 + p_2^2 + 2mU - U^2/c^2) \\ & + (p_1^2 + p_2^2 + 2mU - U^2/c^2)^2. \end{aligned} \quad (18)$$

According to (18), the relativistic Hamiltonian of two-particle systems can be expressed as

$$\begin{aligned} E' = & \frac{2(m_{01}\mu + m_{02}\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{p_1^2}{m_{01}} + \frac{2(m_{02}\mu + m_{01}\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{p_2^2}{m_{02}} \\ & + \frac{2\mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m}U - \frac{U^2}{(m_0 + m)c^2} \right) \\ & - \frac{1}{(m_0 + m)(\mu_0 + \mu)c^2} (p_1^2 + p_2^2) \left( \frac{2m}{m_0 + m}U - \frac{U^2}{(m_0 + m)c^2} \right) \\ & - \frac{1}{(m_0 + m)(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m}U - \frac{U^2}{(m_0 + m)c^2} \right) (p_1^2 + p_2^2) \\ & - \frac{1}{(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m}U - \frac{U^2}{(m_0 + m)c^2} \right)^2 \\ & - \frac{1}{(m_0 + m)^2(\mu_0 + \mu)c^2} (p_1^2 - p_2^2)^2. \end{aligned} \quad (19)$$

Therefore, taking (19) as the characteristic equation, multiplying both sides of the equation by the base function  $\Psi_p(\mathbf{r}, t)$ , and by using (4), we have

$$\begin{aligned} i\hbar \frac{\partial \Psi_p}{\partial t} = & - \frac{2(m_{01}\mu + m_{02}\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{\hbar^2}{m_{01}} \nabla_1^2 \Psi_p - \frac{2(m_{02}\mu + m_{01}\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{\hbar^2}{m_{02}} \nabla_2^2 \Psi_p \\ & + \frac{2\mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m}U - \frac{U^2}{(m_0 + m)c^2} \right) \Psi_p \\ & + \frac{\hbar^2}{(m_0 + m)(\mu_0 + \mu)c^2} (\nabla_1^2 + \nabla_2^2) \left[ \left( \frac{2m}{m_0 + m}U - \frac{U^2}{(m_0 + m)c^2} \right) \Psi_p \right] \\ & + \frac{\hbar^2}{(m_0 + m)(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m}U - \frac{U^2}{(m_0 + m)c^2} \right) (\nabla_1^2 + \nabla_2^2) \Psi_p \\ & - \frac{1}{(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m}U - \frac{U^2}{(m_0 + m)c^2} \right)^2 \Psi_p \\ & - \frac{\hbar^4}{(m_0 + m)^2(\mu_0 + \mu)c^2} (\nabla_1^2 - \nabla_2^2)^2 \Psi_p + m_0 c^2 \Psi_p. \end{aligned}$$

Where  $U(\mathbf{r}_1, \mathbf{r}_2)$  denotes the potential energy of the interaction between two particles,  $\nabla_1^2, \nabla_2^2$  are Laplace operators respectively corresponding to  $\mathbf{r}_1, \mathbf{r}_2$ .

According to (2), in the operator equation which is tenable for the base function  $\Psi_p(\mathbf{r}, t)$ , as long as each operator in the operator equation is a linear operator and each linear operator does not explicitly contain the characters  $E$  and  $\mathbf{p}$  of  $\Psi_p(\mathbf{r}, t)$ , then this operator equation is also tenable for an arbitrary wave function  $\Psi(\mathbf{r}, t)$ . Whereas, considering that the system mass  $m$  is equivalent to the character  $E$ , this operator equation is not tenable for arbitrary wave functions, but tenable for an stationary wave function like (6). Thus we have:

An isolated two-particle system, the total spin angular momentum of which is zero, is described by the stationary wave function  $\Psi(\mathbf{r}_1, \mathbf{r}_2, t)$  or  $\psi(\mathbf{r}_1, \mathbf{r}_2)$ , any stationary wave function

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, t) = \psi(\mathbf{r}_1, \mathbf{r}_2) \exp(-iEt/\hbar)$$

satisfies the following relativistic wave equation

$$\begin{aligned} i\hbar \frac{\partial \Psi}{\partial t} = & -\frac{2(m_{01}\mu + m_{02}\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{\hbar^2}{m_{01}} \nabla_1^2 \Psi - \frac{2(m_{02}\mu + m_{01}\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{\hbar^2}{m_{02}} \nabla_2^2 \Psi \\ & + \frac{2\mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \Psi \\ & - \frac{1}{(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right)^2 \Psi \\ & + \frac{\hbar^2}{(m_0 + m)(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) (\nabla_1^2 + \nabla_2^2) \Psi \\ & + \frac{\hbar^2}{(m_0 + m)(\mu_0 + \mu)c^2} (\nabla_1^2 + \nabla_2^2) \left[ \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \Psi \right] \\ & - \frac{\hbar^4}{(m_0 + m)^2(\mu_0 + \mu)c^2} (\nabla_1^2 - \nabla_2^2)^2 \Psi + m_0 c^2 \Psi. \end{aligned} \quad (20)$$

Where  $m_0 = m_{01} + m_{02}$ ,  $E' = E - m_0 c^2$ ,  $E = m c^2$ .  $m, \mu_0, \mu$  respectively denote

$$m = m_0 + \frac{1}{c^2} E', \quad \mu_0 = \frac{2m_{01}m_{02}}{m_0 + m}, \quad \mu = \mu_0 + \frac{1}{c^2} E'.$$

Clearly, for non-relativistic limits, we have

$$\mu \rightarrow \mu_0 \rightarrow \frac{m_{01}m_{02}}{m_0} = \frac{m_{01}m_{02}}{m_{01} + m_{02}}.$$

In other words, the relativistic wave function  $\psi(\mathbf{r}_1, \mathbf{r}_2)$  for two-particle systems is determined by the following relativistic wave equation and natural boundary conditions:

$$\begin{aligned}
E'\psi = & -\frac{2(m_{01}\mu + m_{02}\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{\hbar^2}{m_{01}} \nabla_1^2 \psi - \frac{2(m_{02}\mu + m_{01}\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{\hbar^2}{m_{02}} \nabla_2^2 \psi \\
& + \frac{2\mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \psi \\
& - \frac{1}{(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right)^2 \psi \\
& + \frac{\hbar^2}{(m_0 + m)(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) (\nabla_1^2 + \nabla_2^2) \psi \\
& + \frac{\hbar^2}{(m_0 + m)(\mu_0 + \mu)c^2} (\nabla_1^2 + \nabla_2^2) \left[ \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \psi \right] \\
& - \frac{\hbar^4}{(m_0 + m)^2(\mu_0 + \mu)c^2} (\nabla_1^2 - \nabla_2^2)^2 \psi.
\end{aligned} \tag{21}$$

For bound states, the total energy  $E$  of the system is quantized, which is called the system energy level. The system mass  $m = E/c^2$  is uniquely determined by the system energy level  $E$ . Clearly, for non-relativistic limits, this equation turns out to be the Schrödinger equation of two-particle systems. If the system is in the external field, then the system potential energy  $U(\mathbf{r}_1, \mathbf{r}_2)$  includes both the potential energy of the system in the external field and the interaction energy between particles.

Using the center-of-momentum frame, then according to (8) we have  $p_1^2 = p_2^2 = p^2$ , where  $p$  is the relative momentum. Considering

$$\frac{2(m_{01}\mu + m_{02}\mu_0)}{(m_0 + m)(\mu_0 + \mu)m_{01}} + \frac{2(m_{02}\mu + m_{01}\mu_0)}{(m_0 + m)(\mu_0 + \mu)m_{02}} = \frac{4m^2}{(\mu_0 + \mu)(m_0 + m)^2}, \tag{22}$$

then in the center-of-momentum frame, (19) is simplified as

$$\begin{aligned}
E' = & \left( \frac{2m}{m_0 + m} \right)^2 \frac{p^2}{\mu_0 + \mu} + \frac{2\mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \\
& - \frac{2}{(m_0 + m)c^2} \frac{p^2}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \\
& - \frac{2}{(m_0 + m)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \frac{p^2}{\mu_0 + \mu} \\
& - \frac{1}{(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right)^2.
\end{aligned} \tag{23}$$

Taking (23) as the characteristic equation, similarly we have: Considering an isolated two-particle system in the center-of-momentum frame, if the total spin angular momentum

of the system is zero, then the stationary wave function

$$\Psi(\mathbf{r}, t) = \psi(\mathbf{r}) \exp(-iEt/\hbar)$$

is determined by the following relativistic wave equation and natural boundary conditions:

$$\begin{aligned} i\hbar \frac{\partial \Psi}{\partial t} = & - \left( \frac{2m}{m_0 + m} \right)^2 \frac{\hbar^2}{\mu_0 + \mu} \nabla^2 \Psi + \frac{2\mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \Psi \\ & + \frac{2}{(m_0 + m)c^2} \frac{\hbar^2}{\mu_0 + \mu} \nabla^2 \left[ \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \Psi \right] \\ & + \frac{2}{(m_0 + m)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \frac{\hbar^2}{\mu_0 + \mu} \nabla^2 \Psi \\ & - \frac{1}{(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right)^2 \Psi + m_0 c^2 \Psi. \end{aligned} \quad (24)$$

$$\begin{aligned} E' \psi = & - \left( \frac{2m}{m_0 + m} \right)^2 \frac{\hbar^2}{\mu_0 + \mu} \nabla^2 \psi + \frac{2\mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \psi \\ & + \frac{2}{(m_0 + m)c^2} \frac{\hbar^2}{\mu_0 + \mu} \nabla^2 \left[ \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \psi \right] \\ & + \frac{2}{(m_0 + m)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \frac{\hbar^2}{\mu_0 + \mu} \nabla^2 \psi \\ & - \frac{1}{(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right)^2 \psi. \end{aligned} \quad (25)$$

These are also the expressions of relativistic wave equations (20) and (21) in the center-of-momentum frame respectively, where  $\nabla^2$  is the Laplace operator corresponding to the relative coordinate  $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ .

Relativistic wave equations (24) and (25) can be further expressed as

$$\begin{aligned} i\hbar \frac{\partial \Psi}{\partial t} = & - \left( \frac{2m}{m_0 + m} \right)^2 \frac{\hbar^2}{\mu_0 + \mu} \nabla^2 \Psi + \frac{2\mu}{(\mu_0 + \mu)(m_0 + m)} \left( 2mU - \frac{U^2}{c^2} \right) \Psi \\ & + \frac{2\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left[ \nabla^2 \left( 2mU - \frac{U^2}{c^2} \right) \right] \Psi \\ & + \frac{4\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \nabla \left( 2mU - \frac{U^2}{c^2} \right) \cdot \nabla \Psi \\ & + \frac{4\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( 2mU - \frac{U^2}{c^2} \right) \nabla^2 \Psi \\ & - \frac{1}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( 2mU - \frac{U^2}{c^2} \right)^2 \Psi + m_0 c^2 \Psi. \end{aligned} \quad (26)$$

$$\begin{aligned}
E'\psi &= - \left( \frac{2m}{m_0 + m} \right)^2 \frac{\hbar^2}{\mu_0 + \mu} \nabla^2 \psi + \frac{2\mu}{(\mu_0 + \mu)(m_0 + m)} \left( 2mU - \frac{U^2}{c^2} \right) \psi \\
&+ \frac{2\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left[ \nabla^2 \left( 2mU - \frac{U^2}{c^2} \right) \right] \psi \\
&+ \frac{4\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \nabla \left( 2mU - \frac{U^2}{c^2} \right) \cdot \nabla \psi \\
&+ \frac{4\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( 2mU - \frac{U^2}{c^2} \right) \nabla^2 \psi \\
&- \frac{1}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( 2mU - \frac{U^2}{c^2} \right)^2 \psi.
\end{aligned} \tag{27}$$

If  $\Psi(\mathbf{r}, t)$  is the solution of (26), then for  $w(\mathbf{r}, t) = \Psi^*(\mathbf{r}, t)\Psi(\mathbf{r}, t)$  we have

$$\begin{aligned}
\frac{\partial w}{\partial t} &= \left( \frac{2m}{m_0 + m} \right)^2 \frac{i\hbar}{\mu_0 + \mu} (\Psi^* \nabla^2 \Psi - \Psi \nabla^2 \Psi^*) \\
&+ \frac{1}{i\hbar} \frac{4\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \nabla \left( 2mU - \frac{U^2}{c^2} \right) \cdot (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) \\
&+ \frac{1}{i\hbar} \frac{4\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( 2mU - \frac{U^2}{c^2} \right) (\Psi^* \nabla^2 \Psi - \Psi \nabla^2 \Psi^*) \\
&= \left( \frac{2m}{m_0 + m} \right)^2 \frac{i\hbar}{\mu_0 + \mu} \nabla \cdot (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) \\
&- \frac{4i\hbar}{(m_0 + m)^2(\mu_0 + \mu)c^2} \nabla \left( 2mU - \frac{U^2}{c^2} \right) \cdot (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) \\
&- \frac{4i\hbar}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( 2mU - \frac{U^2}{c^2} \right) \nabla \cdot (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) \\
&= \left( \frac{2m}{m_0 + m} \right)^2 \frac{i\hbar}{\mu_0 + \mu} \nabla \cdot (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) \\
&- \frac{4i\hbar}{(m_0 + m)^2(\mu_0 + \mu)c^2} \nabla \cdot \left[ \left( 2mU - \frac{U^2}{c^2} \right) (\Psi^* \nabla \Psi - \Psi \nabla \Psi^*) \right].
\end{aligned}$$

In other words, according to (26),  $w(\mathbf{r}, t) = \Psi^*(\mathbf{r}, t)\Psi(\mathbf{r}, t)$  is defined as the probability density, then the corresponding probability current density vector  $\mathbf{J}$  is expressed as

$$\begin{aligned}
\mathbf{J} &= \left( \frac{2m}{m_0 + m} \right)^2 \frac{i\hbar}{\mu_0 + \mu} (\Psi \nabla \Psi^* - \Psi^* \nabla \Psi) \\
&- \frac{4i\hbar}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( 2mU - \frac{U^2}{c^2} \right) (\Psi \nabla \Psi^* - \Psi^* \nabla \Psi) \\
&= \frac{4i\hbar}{(m_0 + m)^2(\mu_0 + \mu)} \left( m^2 - 2m\frac{U}{c^2} + \frac{U^2}{c^4} \right) (\Psi \nabla \Psi^* - \Psi^* \nabla \Psi).
\end{aligned}$$

Therefore, similar to non-relativistic quantum mechanics, for a two-particle system that the total spin angular momentum of the system is zero, we have

$$\frac{\partial w}{\partial t} + \nabla \cdot \mathbf{J} = 0. \quad (28)$$

Where  $\mathbf{J}$  is expressed as

$$\mathbf{J} = \frac{4i\hbar}{(m_0 + m)^2(\mu_0 + \mu)} \left( m - \frac{U}{c^2} \right)^2 (\Psi \nabla \Psi^* - \Psi^* \nabla \Psi). \quad (29)$$

On the right-hand side of (27), if combining the first and the fifth terms, the stationary relativistic wave equation for two-particle systems in the center-of-momentum frame is expressed as

$$\begin{aligned} E' \psi = & - \frac{4\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)} \left( m - \frac{U}{c^2} \right)^2 \nabla^2 \psi \\ & + \frac{2\mu}{(\mu_0 + \mu)(m_0 + m)} \left( 2mU - \frac{U^2}{c^2} \right) \psi \\ & + \frac{2\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left[ \nabla^2 \left( 2mU - \frac{U^2}{c^2} \right) \right] \psi \\ & + \frac{4\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \nabla \left( 2mU - \frac{U^2}{c^2} \right) \cdot \nabla \psi \\ & - \frac{1}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( 2mU - \frac{U^2}{c^2} \right)^2 \psi. \end{aligned} \quad (30)$$

### 3. Relativistic Two-body Wave Equations and Spin Angular Momentum

As we know, the spin angular momentum not only has the general property of angular momentum  $\mathbf{S} \times \mathbf{S} = i\hbar\mathbf{S}$ , but also has its own particularity. For instance, for electrons and protons, the projection of the spin angular momentum  $\mathbf{S}$  in any direction only takes two values  $\pm\hbar/2$ .

For the convenience of studying this type of angular momentum, a type of dimensionless vector  $\sigma$  is introduced, determined by

$$\sigma \times \sigma = 2i\sigma, \quad \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1. \quad (31)$$

Using this type of vector  $\sigma$ , the spin angular momentum is expressed as  $\mathbf{S} = (\hbar/2)\sigma$ . According to (31), if  $\mathbf{A}$ ,  $\mathbf{B}$  are two arbitrary vectors commuted with  $\sigma$ , we have

$$(\sigma \cdot \mathbf{A})(\sigma \cdot \mathbf{B}) = \mathbf{A} \cdot \mathbf{B} + i\sigma \cdot (\mathbf{A} \times \mathbf{B}). \quad (32)$$

If the two particles composing the system both have spin 1/2 angular momentum, denoted by  $\mathbf{S}_1$  and  $\mathbf{S}_2$ :

$$\mathbf{S}_1 = \frac{\hbar}{2}\sigma_1, \quad \mathbf{S}_2 = \frac{\hbar}{2}\sigma_2,$$

then the relativistic wave function describing the system should be a spinor. Let  $\phi, \psi$  be arbitrary spinors, then we can easily prove:

$$(\sigma \cdot \mathbf{p})(\phi^\dagger \psi) = \phi^\dagger(\sigma \cdot \mathbf{p})\psi + (\sigma \cdot \mathbf{p}\phi^\dagger)\psi. \quad (33)$$

Repeatedly using (33), considering  $(\sigma \cdot \mathbf{p})(\sigma \cdot \mathbf{p}) = \mathbf{p} \cdot \mathbf{p} = p^2$ , we have

$$p^2(\phi^\dagger \psi) = \phi^\dagger p^2 \psi + 2(\sigma \cdot \mathbf{p}\phi^\dagger)(\sigma \cdot \mathbf{p}\psi) + (p^2 \phi^\dagger)\psi. \quad (34)$$

Therefore, under the condition that wave functions are spinors, the relativistic Hamiltonian of two-particle systems (19) is rewritten as

$$\begin{aligned} E' = & \frac{2(m_{01}\mu + m_{02}\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{p_1^2}{m_{01}} + \frac{2(m_{02}\mu + m_{01}\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{p_2^2}{m_{02}} \\ & + \frac{2\mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \\ & - \frac{2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left[ \sigma_1 \cdot \mathbf{p}_1 \left( 2mU - \frac{U^2}{c^2} \right) \right] (\sigma_1 \cdot \mathbf{p}_1) \\ & - \frac{2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left[ \sigma_2 \cdot \mathbf{p}_2 \left( 2mU - \frac{U^2}{c^2} \right) \right] (\sigma_2 \cdot \mathbf{p}_2) \\ & - \frac{2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( 2mU - \frac{U^2}{c^2} \right) (p_1^2 + p_2^2) \\ & - \frac{1}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left[ (p_1^2 + p_2^2) \left( 2mU - \frac{U^2}{c^2} \right) \right] \\ & - \frac{1}{(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right)^2 \\ & - \frac{1}{(m_0 + m)^2(\mu_0 + \mu)c^2} (p_1^2 - p_2^2)^2. \end{aligned} \quad (35)$$

Thus the stationary relativistic wave equation for two-particle systems can be expressed as

$$\begin{aligned}
i\hbar \frac{\partial \Psi}{\partial t} = & -\frac{2(m_{01}\mu + m_{02}\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{\hbar^2}{m_{01}} \nabla_1^2 \Psi - \frac{2(m_{02}\mu + m_{01}\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{\hbar^2}{m_{02}} \nabla_2^2 \Psi \\
& + \frac{2\mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \Psi \\
& + \frac{2\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left[ \sigma_1 \cdot \nabla_1 \left( 2mU - \frac{U^2}{c^2} \right) \right] (\sigma_1 \cdot \nabla_1) \Psi \\
& + \frac{2\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left[ \sigma_2 \cdot \nabla_2 \left( 2mU - \frac{U^2}{c^2} \right) \right] (\sigma_2 \cdot \nabla_2) \Psi \\
& + \frac{2\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( 2mU - \frac{U^2}{c^2} \right) (\nabla_1^2 + \nabla_2^2) \Psi \\
& + \frac{\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left[ (\nabla_1^2 + \nabla_2^2) \left( 2mU - \frac{U^2}{c^2} \right) \right] \Psi \\
& - \frac{1}{(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right)^2 \Psi \\
& - \frac{\hbar^4}{(m_0 + m)^2(\mu_0 + \mu)c^2} (\nabla_1^2 - \nabla_2^2)^2 \Psi + m_0 c^2 \Psi. \tag{36}
\end{aligned}$$

The two particles of the system are spin 1/2 particles, where  $U(\mathbf{r}_1, \mathbf{r}_2)$  denotes the potential energy of the system in the external field and the interaction energy between particles.

In the central field, the potential energy  $U(\mathbf{r})$  is only related to the distance between particles, but not related to the direction of  $\mathbf{r}$ . According to (32), for the fourth and the fifth terms on the right-hand side of (36), we have

$$\begin{aligned}
& \frac{2\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left[ \sigma \cdot \nabla \left( 2mU - \frac{U^2}{c^2} \right) \right] (\sigma \cdot \nabla \Psi) \\
= & \frac{2\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \nabla \left( 2mU - \frac{U^2}{c^2} \right) \cdot \nabla \Psi \\
& + \frac{2\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} i\sigma \cdot \left[ \nabla \left( 2mU - \frac{U^2}{c^2} \right) \times \nabla \Psi \right] \\
= & \frac{4\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( m - \frac{U}{c^2} \right) \frac{dU}{dr} \frac{\partial \Psi}{\partial r} \\
& - \frac{8}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( m - \frac{U}{c^2} \right) \frac{1}{r} \frac{dU}{dr} \mathbf{S} \cdot \mathbf{L} \Psi.
\end{aligned}$$

Thus in the central field we have

For spin 1/2 particles, the stationary wave equation for two-particle systems can be express as



$$\begin{aligned}
i\hbar \frac{\partial \Psi}{\partial t} = & -\frac{2(m_{01}\mu + m_{02}\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{\hbar^2}{m_{01}} \nabla_1^2 \Psi - \frac{2(m_{02}\mu + m_{01}\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{\hbar^2}{m_{02}} \nabla_2^2 \Psi \\
& + \frac{2\mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \Psi \\
& + \frac{4\hbar^2(m - U/c^2)}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( \frac{dU}{dr_1} \frac{\partial \Psi}{\partial r_1} + \frac{dU}{dr_2} \frac{\partial \Psi}{\partial r_2} \right) \\
& + \frac{2\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( 2mU - \frac{U^2}{c^2} \right) (\nabla_1^2 + \nabla_2^2) \Psi \\
& + \frac{\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left[ (\nabla_1^2 + \nabla_2^2) \left( 2mU - \frac{U^2}{c^2} \right) \right] \Psi \\
& - \frac{1}{(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right)^2 \Psi \\
& - \frac{8(m - U/c^2)}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( \frac{1}{r_1} \frac{dU}{dr_1} \mathbf{S}_1 \cdot \mathbf{L}_1 \Psi + \frac{1}{r_2} \frac{dU}{dr_2} \mathbf{S}_2 \cdot \mathbf{L}_2 \Psi \right) \\
& - \frac{\hbar^4}{(m_0 + m)^2(\mu_0 + \mu)c^2} (\nabla_1^2 - \nabla_2^2)^2 \Psi + m_0 c^2 \Psi. \tag{37}
\end{aligned}$$

$$\begin{aligned}
E' \psi = & -\frac{2(m_{01}\mu + m_{02}\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{\hbar^2}{m_{01}} \nabla_1^2 \psi - \frac{2(m_{02}\mu + m_{01}\mu_0)}{(m_0 + m)(\mu_0 + \mu)} \frac{\hbar^2}{m_{02}} \nabla_2^2 \psi \\
& + \frac{2\mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \psi \\
& + \frac{4\hbar^2(m - U/c^2)}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( \frac{dU}{dr_1} \frac{\partial \psi}{\partial r_1} + \frac{dU}{dr_2} \frac{\partial \psi}{\partial r_2} \right) \\
& + \frac{2\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( 2mU - \frac{U^2}{c^2} \right) (\nabla_1^2 + \nabla_2^2) \psi \\
& + \frac{\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left[ (\nabla_1^2 + \nabla_2^2) \left( 2mU - \frac{U^2}{c^2} \right) \right] \psi \\
& - \frac{1}{(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right)^2 \psi \\
& - \frac{8(m - U/c^2)}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( \frac{1}{r_1} \frac{dU}{dr_1} \mathbf{S}_1 \cdot \mathbf{L}_1 \psi + \frac{1}{r_2} \frac{dU}{dr_2} \mathbf{S}_2 \cdot \mathbf{L}_2 \psi \right) \\
& - \frac{\hbar^4}{(m_0 + m)^2(\mu_0 + \mu)c^2} (\nabla_1^2 - \nabla_2^2)^2 \psi. \tag{38}
\end{aligned}$$

Where  $\mathbf{L}_1$  is the orbital angular momentum of the first particle, and  $\mathbf{L}_2$  is that of the second one.  $\Psi$  is the stationary relativistic wave function

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, s_{1z}, s_{2z}, t) = \psi(\mathbf{r}_1, \mathbf{r}_2, s_{1z}, s_{2z}) \exp(-iEt/\hbar).$$

For an isolated two-particle system, in the center-of-momentum frame,  $\mathbf{p}_2 = -\mathbf{p}_1$ ,  $|\mathbf{p}_1| = |\mathbf{p}|$ , and  $|\mathbf{p}|$  is the relative momentum of the two-particle system. Based on the corresponding relation between momentum operators and gradient operators, according to  $\mathbf{p}_2 = -\mathbf{p}_1 = -\mathbf{p}$  we have  $\nabla_2 = -\nabla_1 = -\nabla$ . Where  $\nabla_1$ ,  $\nabla_2$  and  $\nabla$  are gradient operators corresponding to the coordinates  $\mathbf{r}_1$ ,  $\mathbf{r}_2$  and  $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ .

Therefore, if using the center-of-momentum frame in the wave equation (37), supposing an isolated two-particle system, then Using  $\nabla_2 = -\nabla_1 = -\nabla$  and  $\mathbf{p}_2 = -\mathbf{p}_1 = -\mathbf{p}$ , in the central field we have

$$\begin{aligned}\frac{dU}{dr_1} \frac{\partial \Psi}{\partial r_1} &= (\nabla_1 U) \cdot \nabla_1 \Psi = (\nabla U) \cdot \nabla \Psi = \frac{dU}{dr} \frac{\partial \Psi}{\partial r} \\\frac{dU}{dr_2} \frac{\partial \Psi}{\partial r_2} &= (\nabla_2 U) \cdot \nabla_2 \Psi = (-\nabla U) \cdot (-\nabla \Psi) = \frac{dU}{dr} \frac{\partial \Psi}{\partial r} \\\frac{1}{r_1} \frac{dU}{dr_1} \mathbf{L}_1 &= \frac{1}{r_1} \frac{dU}{dr_1} \mathbf{r}_1 \times \mathbf{p}_1 = (\nabla_1 U) \times \mathbf{p}_1 = (\nabla U) \times \mathbf{p} = \frac{1}{r} \frac{dU}{dr} \mathbf{r} \times \mathbf{p} \\\frac{1}{r_2} \frac{dU}{dr_2} \mathbf{L}_2 &= \frac{1}{r_2} \frac{dU}{dr_2} \mathbf{r}_2 \times \mathbf{p}_2 = (\nabla_2 U) \times \mathbf{p}_2 = (-\nabla U) \times (-\mathbf{p}) = \frac{1}{r} \frac{dU}{dr} \mathbf{r} \times \mathbf{p}.\end{aligned}$$

Considering (22), the wave equation (37) can be expressed as

$$\begin{aligned}i\hbar \frac{\partial \Psi}{\partial t} &= -\frac{4m^2\hbar^2}{(\mu_0 + \mu)(m_0 + m)^2} \nabla^2 \Psi + \frac{2\mu}{\mu_0 + \mu} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right) \Psi \\&+ \frac{8\hbar^2(m - U/c^2)}{(m_0 + m)^2(\mu_0 + \mu)c^2} \frac{dU}{dr} \frac{\partial \Psi}{\partial r} \\&+ \frac{4\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left( 2mU - \frac{U^2}{c^2} \right) \nabla^2 \Psi \\&+ \frac{2\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left[ \nabla^2 \left( 2mU - \frac{U^2}{c^2} \right) \right] \Psi \\&- \frac{1}{(\mu_0 + \mu)c^2} \left( \frac{2m}{m_0 + m} U - \frac{U^2}{(m_0 + m)c^2} \right)^2 \Psi \\&- \frac{8(m - U/c^2)}{(m_0 + m)^2(\mu_0 + \mu)c^2} \frac{1}{r} \frac{dU}{dr} (\mathbf{S}_1 + \mathbf{S}_2) \cdot (\mathbf{r} \times \mathbf{p}) \Psi + m_0 c^2 \Psi.\end{aligned}\tag{39}$$

The total spin angular momentum of the system is  $\mathbf{S} = \mathbf{S}_1 + \mathbf{S}_2$ , and the orbital angular momentum  $\mathbf{L}$  is

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 = \mathbf{r}_1 \times \mathbf{p}_1 + \mathbf{r}_2 \times \mathbf{p}_2 = (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{p}_1 = \mathbf{r} \times \mathbf{p}.$$

Therefore, the total orbital angular momentum  $\mathbf{L}$  of the two-particle system in the center-of-momentum frame is equal to the cross product of the relative coordinate  $\mathbf{r}$  and the

relative momentum  $\mathbf{p}$ . Combining the first and the fourth terms on the right-hand side of (39), we have

Let  $\mathbf{S}$ ,  $\mathbf{L}$  be the total spin angular momentum and total orbital angular momentum operators of the two-particle system respectively,  $U(\mathbf{r})$  be the interaction energy between particles, then in the central field, the stationary wave function for the two-particle system in the center-of-momentum frame

$$\Psi(\mathbf{r}, s_z, t) = \psi(\mathbf{r}, s_z) \exp(-iEt/\hbar)$$

is determined by the following relativistic wave function and natural boundary conditions, namely

$$\begin{aligned} i\hbar \frac{\partial \Psi}{\partial t} = & -\frac{4\hbar^2}{(\mu_0 + \mu)(m_0 + m)^2} \left(m - \frac{U}{c^2}\right)^2 \nabla^2 \Psi \\ & + \frac{2\mu}{(\mu_0 + \mu)(m_0 + m)} \left(2mU - \frac{U^2}{c^2}\right) \Psi \\ & + \frac{8\hbar^2(m - U/c^2)}{(m_0 + m)^2(\mu_0 + \mu)c^2} \frac{dU}{dr} \frac{\partial \Psi}{\partial r} \\ & + \frac{2\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left[ \nabla^2 \left(2mU - \frac{U^2}{c^2}\right) \right] \Psi \\ & - \frac{1}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left(2mU - \frac{U^2}{c^2}\right)^2 \Psi \\ & - \frac{8(m - U/c^2)}{(m_0 + m)^2(\mu_0 + \mu)c^2} \frac{1}{r} \frac{dU}{dr} \mathbf{S} \cdot \mathbf{L} \Psi + m_0 c^2 \Psi. \end{aligned} \quad (40)$$

$$\begin{aligned} E' \psi = & -\frac{4\hbar^2}{(\mu_0 + \mu)(m_0 + m)^2} \left(m - \frac{U}{c^2}\right)^2 \nabla^2 \psi \\ & + \frac{2\mu}{(\mu_0 + \mu)(m_0 + m)} \left(2mU - \frac{U^2}{c^2}\right) \psi \\ & + \frac{8\hbar^2(m - U/c^2)}{(m_0 + m)^2(\mu_0 + \mu)c^2} \frac{dU}{dr} \frac{\partial \psi}{\partial r} \\ & + \frac{2\hbar^2}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left[ \nabla^2 \left(2mU - \frac{U^2}{c^2}\right) \right] \psi \\ & - \frac{1}{(m_0 + m)^2(\mu_0 + \mu)c^2} \left(2mU - \frac{U^2}{c^2}\right)^2 \psi \\ & - \frac{8(m - U/c^2)}{(m_0 + m)^2(\mu_0 + \mu)c^2} \frac{1}{r} \frac{dU}{dr} \mathbf{S} \cdot \mathbf{L} \psi. \end{aligned} \quad (41)$$

According to the wave equation (30), in the central field, the stationary relativistic wave equation for the two-particle system that the total spin angular momentum is zero in the center-of-momentum frame is expressed as

$$\begin{aligned}
E'\psi = & -\frac{4\hbar^2}{(m_0+m)^2(\mu_0+\mu)}\left(m-\frac{U}{c^2}\right)^2\nabla^2\psi \\
& +\frac{2\mu}{(\mu_0+\mu)(m_0+m)}\left(2mU-\frac{U^2}{c^2}\right)\psi \\
& +\frac{2\hbar^2}{(m_0+m)^2(\mu_0+\mu)c^2}\left[\nabla^2\left(2mU-\frac{U^2}{c^2}\right)\right]\psi \\
& +\frac{8\hbar^2}{(m_0+m)^2(\mu_0+\mu)c^2}\left(m-\frac{U}{c^2}\right)\frac{dU}{dr}\frac{\partial\psi}{\partial r} \\
& -\frac{1}{(m_0+m)^2(\mu_0+\mu)c^2}\left(2mU-\frac{U^2}{c^2}\right)^2\psi.
\end{aligned} \tag{42}$$

Compared with (42), (41) has one more term, corresponding to the coupling energy between the total spin angular momentum and the total orbital angular momentum.

In (35)-(42),  $E' = E - m_0c^2$ ,  $m_0 = m_{01} + m_{02}$ ,  $E = mc^2$ ,  $m, \mu_0, \mu$  respectively denote

$$m = m_0 + \frac{1}{c^2}E', \quad \mu_0 = \frac{2m_{01}m_{02}}{m_0 + m}, \quad \mu = \mu_0 + \frac{1}{c^2}E'.$$

If applying the relativistic wave equation (41) for two-particle systems to hydrogen-like atoms, then  $U/c^2 \ll m$ , (41) can be approximately expressed as

$$\begin{aligned}
E'\psi = & -\frac{4m^2}{(m_0+m)^2}\frac{\hbar^2}{\mu_0+\mu}\nabla^2\psi + \frac{4m\mu U}{(m_0+m)(\mu_0+\mu)}\psi - \frac{4m^2}{(m_0+m)^2}\frac{U^2}{(\mu_0+\mu)c^2}\psi \\
& +\frac{8m}{(m_0+m)^2(\mu_0+\mu)c^2}\left(\hbar^2\frac{dU}{dr}\frac{\partial\psi}{\partial r} - \frac{1}{r}\frac{dU}{dr}\mathbf{S}\cdot\mathbf{L}\psi + \frac{\hbar^2}{2}(\nabla^2U)\psi\right).
\end{aligned} \tag{43}$$

Applying this wave equation, we expect to determine the hyperfine structure of the energy levels of hydrogen-like atoms and the Lamb shift with high precision.

## 4. Relativistic Energy Levels for Two-Particle Systems with Zero Total Spin Angular Momentum

A hydrogen-like atom that the total spin angular momentum is zero, and the ponium composed by  $\pi^-$  and  $\pi^+$ , are both two-particle systems. This type of potential energy of

interaction between particles is  $U = -Ze_s^2/r$ ,  $e_s = e(4\pi\epsilon_0)^{-1/2}$ , considering

$$\nabla^2 \frac{1}{r} = -4\pi\delta(\mathbf{r}),$$

the wave equation (42) can be further expressed as

$$\begin{aligned} E'\psi &= -\frac{4\hbar^2}{(m_0+m)^2(\mu_0+\mu)} \left(m + \frac{Ze_s^2}{c^2r}\right)^2 \nabla^2\psi \\ &\quad - \frac{2\mu}{(\mu_0+\mu)(m_0+m)} \left(\frac{2mZe_s^2}{r} + \frac{Z^2e_s^4}{c^2r^2}\right) \psi \\ &\quad + \frac{8\hbar^2}{(m_0+m)^2(\mu_0+\mu)c^2} \left(m + \frac{Ze_s^2}{c^2r}\right) \frac{Ze_s^2}{r^2} \frac{\partial\psi}{\partial r} \\ &\quad - \frac{1}{(m_0+m)^2(\mu_0+\mu)c^2} \left(\frac{2mZe_s^2}{r} + \frac{Z^2e_s^4}{c^2r^2}\right)^2 \psi \\ &\quad - \frac{4\hbar^2}{(m_0+m)^2(\mu_0+\mu)c^2} \frac{Z^2e_s^4}{c^2r^4} \psi \\ &\quad + \frac{16\pi m\hbar^2 Ze_s^2}{(m_0+m)^2(\mu_0+\mu)c^2} \delta(\mathbf{r})\psi. \end{aligned} \tag{44}$$

Now let us solve the wave equation (44) under the condition of  $\mathbf{r} > 0$ , when  $\delta(\mathbf{r}) = 0$ . Using the spherical polar coordinates, the Laplace operator  $\nabla^2$  is expressed as

$$\nabla^2 = \frac{1}{r^2} \left[ \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial\varphi^2} \right].$$

Referring to the solving procedures of the Schrödinger equation in Ref. [5], supposing  $\psi(r, \theta, \varphi) = R(r)Y(\theta, \varphi)$ , substituting it into (44) and considering  $\delta(\mathbf{r}) = 0$ , we have

$$\begin{aligned} \frac{(m_0+m)^2(\mu_0+\mu)E'r^2}{4\hbar^2[m + Ze_s^2/(c^2r)]^2} &+ \frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) - \frac{2Ze_s^2}{[m + Ze_s^2/(c^2r)]c^2} \frac{1}{R} \frac{dR}{dr} \\ &+ \frac{\mu(m_0+m)r^2}{2\hbar^2[m + Ze_s^2/(c^2r)]^2} \left( \frac{2mZe_s^2}{r} + \frac{Z^2e_s^4}{c^2r^2} \right) \\ &+ \frac{r^2}{4\hbar^2c^2[m + Ze_s^2/(c^2r)]^2} \left( \frac{2mZe_s^2}{r} + \frac{Z^2e_s^4}{c^2r^2} \right)^2 \\ &+ \frac{1}{[m + Ze_s^2/(c^2r)]^2} \frac{Z^2e_s^4}{c^4r^2} \\ &= -\frac{1}{Y} \left[ \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial Y}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 Y}{\partial\varphi^2} \right] = \lambda. \end{aligned}$$

$$\begin{aligned}
& \left(1 + \frac{Ze_s^2}{mc^2r}\right)^2 \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dR}{dr}\right) - \left(1 + \frac{Ze_s^2}{mc^2r}\right) \frac{2Ze_s^2}{mc^2r^2} \frac{dR}{dr} \\
& + \frac{\mu(m_0 + m)Ze_s^2}{m\hbar^2r} \left(1 + \frac{Ze_s^2}{2mc^2r}\right) R + \frac{(m_0 + m)^2(\mu_0 + \mu)E'}{4m^2\hbar^2} R \\
& + \left(1 + \frac{Ze_s^2}{2mc^2r}\right)^2 \frac{Z^2e_s^4}{\hbar^2c^2r^2} R + \frac{Z^2e_s^4}{m^2c^4r^4} R - \left(1 + \frac{Ze_s^2}{mc^2r}\right)^2 \frac{\lambda}{r^2} R = 0. \tag{45}
\end{aligned}$$

$$\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta} \frac{\partial^2 Y}{\partial\varphi^2} + \lambda Y = 0. \tag{46}$$

According to (46), denoting  $\lambda = l(l+1)$ ,  $l = 0, 1, 2, \dots$ , obviously the solution of the equation is the spherical harmonics  $Y_{lm}(\theta, \varphi)$

Now let us solve the radial equation (45), discussing the situation of the bound state ( $E' < 0$ ). Let

$$\alpha' = \frac{m_0 + m}{m\hbar} [(\mu_0 + \mu)|E'|]^{1/2}, \quad \rho = \alpha' r, \tag{47}$$

$$\beta = \frac{\mu(m_0 + m)Ze_s^2}{\alpha'm\hbar^2} = \frac{Ze_s^2}{\hbar} \left[ \frac{\mu^2}{(\mu_0 + \mu)|E'|} \right]^{1/2}, \tag{48}$$

using the variable substitution  $\rho = \alpha' r$ , then (45) can be expressed as:

$$\begin{aligned}
& \left(1 + \frac{d_0}{\beta\rho}\right)^2 \frac{1}{\rho^2} \frac{d}{d\rho} \left(\rho^2 \frac{dR}{d\rho}\right) - \left(1 + \frac{d_0}{\beta\rho}\right) \frac{2d_0}{\beta\rho^2} \frac{dR}{d\rho} + \frac{\beta}{\rho} \left(1 + \frac{d_0}{2\beta\rho}\right) R \\
& - \frac{1}{4} R + \left(1 + \frac{d_0}{2\beta\rho}\right)^2 \frac{Z^2\alpha^2}{\rho^2} R + \frac{d_0^2}{\beta^2\rho^4} R - \left(1 + \frac{d_0}{\beta\rho}\right)^2 \frac{l(l+1)}{\rho^2} R = 0. \tag{49}
\end{aligned}$$

Where  $\alpha$  denotes the fine structure constant.  $d_0$ , which is a small parameter, denotes

$$d_0 = 2Z^2\alpha^2 D, \quad D = \frac{\mu(m_0 + m)}{2m^2}, \quad \alpha = \frac{e_s^2}{\hbar c}. \tag{50}$$

Let  $R(\rho) = u(\rho)/\rho$ , considering

$$\frac{1}{\rho^2} \frac{d}{d\rho} \left(\rho^2 \frac{dR}{d\rho}\right) = \frac{1}{\rho} \frac{d^2}{d\rho^2} (\rho R),$$

then (49) can be expressed as:

$$\begin{aligned}
& \left(1 + \frac{d_0}{\beta\rho}\right)^2 \frac{d^2 u}{d\rho^2} - \left(1 + \frac{d_0}{\beta\rho}\right) \frac{2d_0}{\beta\rho^2} \frac{du}{d\rho} + \frac{2d_0}{\beta\rho^3} u + \frac{\beta}{\rho} \left(1 + \frac{d_0}{2\beta\rho}\right) u \\
& - \frac{1}{4} u + \left(1 + \frac{d_0}{2\beta\rho}\right)^2 \frac{Z^2\alpha^2}{\rho^2} u + \frac{3d_0^2}{\beta^2\rho^4} u - \left(1 + \frac{d_0}{\beta\rho}\right)^2 \frac{l(l+1)}{\rho^2} u = 0. \tag{51}
\end{aligned}$$

Firstly, let us study the asymptotic behavior of this equation, when  $\rho \rightarrow \infty$ , the equation can be transformed into the following form:

$$\frac{d^2 u}{d\rho^2} - \frac{1}{4}u = 0, \quad u(\rho) = \exp(\pm \rho/2).$$

As  $\exp(\rho/2)$  is in conflict with the finite conditions of wave functions, we substitute  $u(\rho) = \exp(-\rho/2)f(\rho)$  into the equation, then we have the equation satisfied by  $f(\rho)$ :

$$\begin{aligned} & \left(1 + \frac{d_0}{\beta\rho}\right)^2 \frac{d^2 f}{d\rho^2} - \left(1 + \frac{d_0}{\beta\rho}\right) \left(1 + \frac{d_0}{\beta\rho} + \frac{2d_0}{\beta\rho^2}\right) \frac{d f}{d\rho} \\ & + \left(\frac{\beta}{\rho} + \frac{d_0}{2\beta\rho}\right) \left(1 + \frac{d_0}{2\beta\rho}\right) f + \left(1 + \frac{2}{\rho} + \frac{d_0}{\beta\rho} + \frac{3d_0}{\beta\rho^2}\right) \frac{d_0}{\beta\rho^2} f \\ & + \left(1 + \frac{d_0}{2\beta\rho}\right)^2 \frac{Z^2 \alpha^2}{\rho^2} f - \left(1 + \frac{d_0}{\beta\rho}\right)^2 \frac{l(l+1)}{\rho^2} f = 0. \end{aligned} \quad (52)$$

Thus solving for the radial wave function  $R(\rho)$  comes down to solving for  $f(\rho)$ , namely

$$R(\rho) = \frac{1}{\rho} \exp\left(-\frac{1}{2}\rho\right) f(\rho), \quad \rho = \frac{2Z}{\beta a_0} r, \quad a_0 = \frac{2m}{m_0 + m} \frac{\hbar^2}{\mu e_s^2}. \quad (53)$$

According to (48) we have

$$\beta^2 = \frac{Z^2 e_s^4}{\hbar^2} \frac{\mu^2}{(\mu_0 + \mu)|E'|}.$$

Substituting  $\mu = \mu_0 - |E'|/c^2$  into the equation above, we have

$$(Z^2 \alpha^2 + \beta^2)|E'|^2 - 2\mu_0 c^2 (Z^2 \alpha^2 + \beta^2)|E'| + \mu_0^2 c^4 Z^2 \alpha^2 = 0. \quad (54)$$

Solving (54), we obtain two roots of  $|E'|$ :

$$|E'| = \mu_0 c^2 \mp \mu_0 c^2 \left(1 + \frac{Z^2 \alpha^2}{\beta^2}\right)^{-1/2}.$$

Thus we obtain the system mass  $\mu$  corresponding to the reduced mass  $\mu_0$ :

$$\mu = \pm \mu_0 \left(1 + \frac{Z^2 \alpha^2}{\beta^2}\right)^{-1/2}. \quad (55)$$

According to Definition 6,  $\mu_0$ ,  $\mu$  respectively denote

$$\mu_0 = \frac{2m_{01}m_{02}}{m_0 + m}, \quad \mu = \mu_0 - \frac{1}{c^2}|E'|.$$

Considering  $m = m_0 - |E'|/c^2$ ,  $m_0 = m_{01} + m_{02}$ ,  $|E'|$  is derived by (55). As the total energy of the system  $E = m_0 c^2 - |E'| = m c^2$ , we can further obtain the total energy  $E$  and the system mass  $m$ . According to (55),  $\mu$  has positive and negative values. When  $\mu$  takes on positive values,  $E$  is expressed as

$$E = \pm \left[ m_{01}^2 + 2m_{01}m_{02} \left( 1 + \frac{Z^2 \alpha^2}{\beta^2} \right)^{-1/2} + m_{02}^2 \right]^{1/2} c^2.$$

When  $\mu$  takes on negative values,  $E$  is expressed as

$$E = \pm \left[ m_{01}^2 - 2m_{01}m_{02} \left( 1 + \frac{Z^2 \alpha^2}{\beta^2} \right)^{-1/2} + m_{02}^2 \right]^{1/2} c^2.$$

Therefore we have two positive and two negative energy solutions. Negative energy solutions are related to the ubiquity of antimatter, which will not be discussed in this paper. Taking on positive energy solutions, we have

$$E = \left[ m_{01}^2 + \frac{2m_{01}m_{02}}{\sqrt{1 + Z^2 \alpha^2 / \beta^2}} + m_{02}^2 \right]^{1/2} c^2. \quad (56)$$

$$E = \left[ m_{01}^2 - \frac{2m_{01}m_{02}}{\sqrt{1 + Z^2 \alpha^2 / \beta^2}} + m_{02}^2 \right]^{1/2} c^2. \quad (57)$$

Corresponding to the two positive energy solutions, the system mass  $m$  has two expressions:

$$m = \left[ m_{01}^2 + \frac{2m_{01}m_{02}}{\sqrt{1 + Z^2 \alpha^2 / \beta^2}} + m_{02}^2 \right]^{1/2}. \quad (58)$$

$$m = \left[ m_{01}^2 - \frac{2m_{01}m_{02}}{\sqrt{1 + Z^2 \alpha^2 / \beta^2}} + m_{02}^2 \right]^{1/2}. \quad (59)$$

Clearly, when (55) takes on positive values, the system mass  $m$  is expressed by (58). But when (55) takes on negative values,  $m$  is expressed by (59).

In (56) and (57), the quantization of  $E$  is mirrored by the fact that  $\beta$  is related to both the principal quantum number  $n$  and the angular quantum number  $l$ , i.e.  $\beta = \beta(n, l)$ . Solving the equation (52) we obtain the expression of  $\beta(n, l)$ . Therefore (56) and (57) are the general expressions of the relativistic energy levels for two-particle systems.

It seems difficult to accurately solve (52). Let us solve this equation for approximate solutions to obtain the approximate expression of  $\beta(n, l)$ . First, (52) is expressed by the standard form of second-order ordinary differential equations

$$f'' + p(\rho)f' + q(\rho)f = 0.$$



Where  $p(\rho)$ ,  $q(\rho)$  respectively denote

$$p(\rho) = - \left(1 + \frac{d_0}{\beta\rho}\right)^{-1} \left(1 + \frac{d_0}{\beta\rho} + \frac{2d_0}{\beta\rho^2}\right),$$

$$q(\rho) = \left(1 + \frac{d_0}{\beta\rho}\right)^{-2} \left[ \left(\beta + \frac{d_0}{2\beta}\right) \frac{1}{\rho} + \left(Z^2\alpha^2 - l(l+1) + \frac{d_0}{2} + \frac{d_0}{\beta} + \frac{d_0^2}{4\beta^2}\right) \frac{1}{\rho^2} \right]$$

$$+ \left(1 + \frac{d_0}{\beta\rho}\right)^{-2} \left( Z^2\alpha^2 - 2l(l+1) + 2 + \frac{d_0}{\beta} \right) \frac{d_0}{\beta} \frac{1}{\rho^3}$$

$$+ \left(1 + \frac{d_0}{\beta\rho}\right)^{-2} (Z^2\alpha^2 - 4l(l+1) + 12) \frac{d_0^2}{4\beta^2} \frac{1}{\rho^4}.$$

Considering  $d_0$  is very small, we have  $p(\rho) \approx -1$ , and  $q(\rho)$  is approximately expressed as

$$q(\rho) \approx \left(\beta + \frac{d_0}{2\beta}\right) \frac{1}{\rho} + \left(Z^2\alpha^2 - l(l+1) - \frac{3d_0}{2} + \frac{d_0}{\beta}\right) \frac{1}{\rho^2}.$$

(52) is approximately expressed as

$$\frac{d^2 f}{d\rho^2} - \frac{df}{d\rho} + \left[ \left(\beta + \frac{d_0}{2\beta}\right) \frac{1}{\rho} + \left(Z^2\alpha^2 - l(l+1) - \frac{3d_0}{2} + \frac{d_0}{\beta}\right) \frac{1}{\rho^2} \right] f = 0. \quad (60)$$

Thus  $\rho = 0$  is a regular singular point of the equation (60). Suppose the series solution of this equation can be expressed as

$$f(\rho) = \sum_{\nu=0}^{\infty} b_{\nu} \rho^{s+\nu}, \quad b_0 \neq 0. \quad (61)$$

In order to guarantee the finiteness of  $R = u/\rho$  at  $\rho = 0$ ,  $s$  should be no less than 1. By substituting (61) into (60), as the coefficient of  $\rho^{s+\nu-1}$  is equal to zero, we have the relation satisfied by  $b_{\nu}$ :

$$b_{\nu+1} = \frac{s + \nu - [\beta + d_0/(2\beta)]}{(s + \nu)(s + \nu + 1) - l(l+1) + Z^2\alpha^2 - 3d_0/2 + d_0/\beta} b_{\nu}. \quad (62)$$

If the series are infinite series, then when  $\nu \rightarrow \infty$  we have  $b_{\nu+1}/b_{\nu} \rightarrow 1/\nu$ . Therefore, when  $\rho \rightarrow \infty$ , the behaviour of the series is the same as that of  $e^{\rho}$ , thus  $f(\rho)$  in (53) tends to infinity when  $\rho \rightarrow \infty$ , which is in conflict with the finite conditions of wave functions. Therefore, the series should only have finite terms. Let  $b_{n_r} \rho^{s+n_r}$  be the highest-order term, then  $b_{n_r+1} = 0$ . By substituting  $\nu = n_r$  into (62) we have  $\beta + d_0/(2\beta) = n_r + s$ . On the other hand, the series starts from  $\nu = 0$ , therefore,  $b_{-1} = 0$ . Substituting  $\nu = -1$  into

(62), considering  $b_0 \neq 0$ , we have  $s(s-1) = l(l+1) - Z^2\alpha^2 + 3d_0/2 - d_0/\beta$ . Denoting  $n = n_r + l + 1$ , then the following set of equations can be solved for  $s$  and  $\beta$ :

$$\begin{cases} s(s-1) = l(l+1) - Z^2\alpha^2 + 3d_0/2 - d_0/\beta \\ \beta + d_0/(2\beta) = n_r + s \\ n = n_r + l + 1 \end{cases} \quad (63)$$

We derive  $s = 1/2 \pm \sqrt{(l+1/2)^2 - Z^2\alpha^2 + 3d_0/2 - d_0/\beta}$ . Considering  $s$  should not be less than 1,  $s = 1/2 + \sqrt{(l+1/2)^2 - Z^2\alpha^2 + 3d_0/2 - d_0/\beta}$ . Thus we obtain a specific expression of  $\beta(n, l)$ :

$$\beta = n - l - \frac{1}{2} - \frac{d_0}{2\beta} + \sqrt{\left(l + \frac{1}{2}\right)^2 - Z^2\alpha^2 + \frac{3d_0}{2} - \frac{d_0}{\beta}} = n - \sigma_l. \quad (64)$$

Where  $\sigma_l = l + 1/2 + d_0/(2\beta) - \sqrt{(l+1/2)^2 - Z^2\alpha^2 + 3d_0/2 - d_0/\beta}$ .

Therefore, the relativistic energy levels for two-particle systems (56)-(57), the system mass (58)-(59) and (55) can be respectively expressed as

$$E_n = \left[ m_{01}^2 + 2m_{01}m_{02} \left( 1 + \frac{Z^2\alpha^2}{(n - \sigma_l)^2} \right)^{-1/2} + m_{02}^2 \right]^{1/2} c^2. \quad (65)$$

$$E_n = \left[ m_{01}^2 - 2m_{01}m_{02} \left( 1 + \frac{Z^2\alpha^2}{(n - \sigma_l)^2} \right)^{-1/2} + m_{02}^2 \right]^{1/2} c^2. \quad (66)$$

$$m = \left[ m_{01}^2 + 2m_{01}m_{02} \left( 1 + \frac{Z^2\alpha^2}{(n - \sigma_l)^2} \right)^{-1/2} + m_{02}^2 \right]^{1/2}. \quad (67)$$

$$m = \left[ m_{01}^2 - 2m_{01}m_{02} \left( 1 + \frac{Z^2\alpha^2}{(n - \sigma_l)^2} \right)^{-1/2} + m_{02}^2 \right]^{1/2}. \quad (68)$$

$$\mu = \pm \mu_0 \left( 1 + \frac{Z^2\alpha^2}{(n - \sigma_l)^2} \right)^{-1/2}, \quad \mu_0 = \frac{2m_{01}m_{02}}{m_0 + m}. \quad (69)$$

$$\sigma_l = l + \frac{1}{2} + \frac{d_0}{2(n - \sigma_l)} - \sqrt{\left(l + \frac{1}{2}\right)^2 - Z^2\alpha^2 + \frac{3d_0}{2} - \frac{d_0}{n - \sigma_l}}. \quad (70)$$

$$d_0 = 2Z^2\alpha^2 D, \quad D = \frac{\mu(m_0 + m)}{2m^2}. \quad (71)$$

Considering a two-particle system, which is a hydrogen-like atom composed by a spin-zero nucleus (like the deuteron) and a  $\pi^-$ , called a pionic hydrogen atom,  $m_{01}$ ,  $m_{02}$  are the rest mass of  $\pi^-$  and nucleus ( $m_{01} \ll m_{02}$ ), then (65) can be expanded as the following fast convergent infinite series

$$\begin{aligned}
E_n = & m_{02}c^2 + m_{01}c^2 \left(1 + \frac{Z^2\alpha^2}{(n - \sigma_l)^2}\right)^{-1/2} \\
& + \frac{1}{2}m_{01}c^2 \frac{m_{01}}{m_{02}} \frac{Z^2\alpha^2}{(n - \sigma_l)^2} \left(1 + \frac{Z^2\alpha^2}{(n - \sigma_l)^2}\right)^{-1} \\
& - \frac{1}{2}m_{01}c^2 \left(\frac{m_{01}}{m_{02}}\right)^2 \frac{Z^2\alpha^2}{(n - \sigma_l)^2} \left(1 + \frac{Z^2\alpha^2}{(n - \sigma_l)^2}\right)^{-3/2} + \dots
\end{aligned}$$

Clearly, we obtain the normal energy levels for two-particle systems. Using (65), we can calculate the energy spectrum of pionic hydrogen atoms more accurately. The energy levels expressed by (66) are called the abnormal energy levels. Unlike the normal energy levels, the abnormal energy levels decrease with increasing the principal quantum number  $n$ .

According to (65)-(71), we need to use iterative methods for calculation. As  $d_0$  is very small, taking  $d_0 = 0$  in the expression of  $\sigma_l$ , we can obtain the zeroth order approximation of  $\sigma_l$  to calculate that of  $m$  and  $\mu$ , then substitute them into  $d_0$  to obtain its zeroth order approximation. Substituting the zeroth-order approximation of  $d_0$  and  $\sigma_l$  into  $\sigma_l$  to calculate its first-order approximation, repeating this process, using the first-order approximation of  $\sigma_l$  to calculate that of  $m$  and  $\mu$ , substitute them into  $d_0$  to obtain its first-order approximation. Substituting the first-order approximation of  $d_0$  and  $\sigma_l$  into  $\sigma_l$  to calculate its second-order approximation, and repeating this process we can calculate the  $n$ th-order approximation of  $\sigma_l$ , further, we can obtain the energy levels  $E_n$  and reach the required accuracy. This calculation process can also be realized by computer programming. Note that (69) means  $\mu$  has both positive and negative values. When calculating normal energy levels,  $\mu > 0$ , the system mass uses (67). When calculating abnormal energy levels,  $\mu < 0$ ,  $m$  uses (68). Then what is the physical meaning of abnormal energy levels?

A bound state composed of a positive and a negative pion is called the pionium. The positronium, pionium, protonium, neutronium, etc., are generally called the particleium. For this type of system, we have  $Z = 1$  and  $m_{01} = m_{02}$ . Therefore, when the pionium is at abnormal energy levels, according to (66) we have

$$\lim_{n \rightarrow \infty} E_n \rightarrow (m_{01}^2 - 2m_{01}m_{02} + m_{02}^2)^{1/2}c^2 = (m_{01} - m_{02})c^2 = 0.$$

What is the physical meaning of this result? According to  $E_n = mc^2$ , we have  $m = 0$ , which means the disappearance of the particle system and the annihilation of a pair of positive and negative pions. In relativistic quantum mechanics, the meaning of the vacuum state should not be restricted to a state that the energy is zero. For any bound state composed of a particle-antiparticle pair, if it is at abnormal energy levels, it is in the

vacuum state. Therefore, the annihilation of a pair of positive and negative pions has two phases. The first one composes the pionium, while the second one is its transition from normal energy level expressed by (65) to abnormal energy levels expressed by (66). If this process produces  $\gamma$  photons, it means a pair of positive and negative pions annihilates into photons. The reverse process is the pair production of positive and negative pions. A reasonable extension of this concept is that after the annihilation of any particle-antiparticle pair, a small percentage of the energy is generally given to the abnormal energy levels of the particleium. Thus this percentage of energy is also quantized, and its energy spectrum is given by the abnormal energy levels of the particleium. For instance, let  $m_\pi$  be the rest mass of  $\pi^+$ , then the abnormal energy levels of the pionium can be expressed as

$$E_n = \sqrt{2}m_\pi c^2 \sqrt{1 - \left(1 + \frac{\alpha^2}{(n - \sigma_l)^2}\right)^{-1/2}}.$$

The non-relativistic approximation of this formula can be easily obtained:

$$E_n = \frac{\alpha m_\pi c^2}{n}, \quad n = 1, 2, \dots$$

which is completely different from the energy spectrum of normal matter expressed by (65). Thus generally, photons emitted by normal matter do not interact with the particleium at abnormal energy levels. Assuming there is all sorts of the particleium at abnormal energy levels everywhere in the universe, they do not absorb photons emitted by celestial bodies because of the huge difference in their energy spectrum. On the other hand, as antiparticles have opposite properties from particles, the particleium is equivalent to a neutral particle with small cross section. When the particleium annihilates and transits to abnormal energy levels, this feature of small cross section is reserved. In other words, there may be full of energy left behind after the annihilation of particle-antiparticle pairs in the universe, each type of energy has its own spectrum and each spectrum is determined by abnormal energy levels of the corresponding particleium. Based on the relativistic mass-energy relation, the particleium at abnormal energy levels has mass and can cause gravitational effects. Apparently, the particleium can transit from higher abnormal energy levels to lower ones with the emission of photons, but this type of photons can hardly reach our observation instruments as they will be easily absorbed by other particleium at abnormal energy levels nearby, and that is why it seems dark. According to observation results of modern astronomy, there exists huge amount of mysterious dark energy and dark matter in the universe. If the particleium at abnormal energy levels is the carrier of dark energy, and part of them aggregate to form dark matter because of gravitational effects, then the abnormal energy levels can establish the theoretical basis of the existence of dark energy and dark matter.

## 5. Conclusion

In conclusion, by introducing Definition 1-6, using (2) and assuming the relativistic kinetic expression is tenable on a wider scale, the relativistic wave equations for two-particle systems are derived, and the new relativistic two-body wave equations are obtained, from which, the spin-orbit coupling term is also derived. By applying this type of wave equations to ponium and pionic hydrogen atoms, the general expression and specific calculation formulas of relativistic energy levels for two-particle systems are derived. Besides, we further find the relativistic abnormal energy levels, thus the pair production and annihilation of particles and antiparticles boil down to the transition between normal and abnormal energy levels of two-particle systems. Then the relation between abnormal energy levels and dark energy is discussed. We believe there are all sorts of particleium at abnormal energy levels everywhere in the universe, which are not only dark, but also hardly interact with celestial bodies and photons passing through them. Proposing the abnormal energy levels paves a new road to reveal the mysteries of dark energy and dark matter.

## References

- [1] G.Q. Bi 1997 *Pure and Applied Mathematics*, **13**(1): 7-14
- [2] G.Q. Bi 1999 *Chinese Quarterly Journal of Mathematics*, **14**(3): 80-87
- [3] G.Q. Bi 2001 *Chinese Quarterly Journal of Mathematics*, **16**(1): 88-101
- [4] G.Q. Bi and Y.K. Bi 2011 *Chinese Quarterly Journal of Mathematics*, **26**(4): 511-515
- [5] S.X. Zhou 1979 *Quantum Mechanics Course*, Beijing: People's Education Press
- [6] D.P. Stanley and D. Robson 1980 *Phys. Rev.* **D21**: 3180
- [7] S. Godfrey and N. Isgur 1985 *Phys. Rev.* **D32**: 189
- [8] G.Q. Bi and Y.K. Bi 2010 arXiv:1008.4224
- [9] G.Q. Bi and Y.K. Bi 2010 arXiv:1008.3808